UNNS Substrate Research Program | Working Manuscript

Admissible Cluster Geometry: Recoverable Connectivity in Realizability Space

Internal Topology of $\mathcal{M}_{\text{adm}}$ from Multi-Domain Structural Corpora

Instruments: STRUC-I v1.0.4 · STRUC-PERC-I v2.4.0–v2.5.0
Corpus: 500 metallic-glass ladders · 67 neutrino-detector ladders (Stage 1 & 2) · 188 normalized spectral ladders (7 grid configurations) · 5 protein MSM ladders
Domains: SciGlass spectral chemistry · Liquid-scintillator detector reconstruction · Protein Markov State Model (Folding@home)
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Abstract

Prior UNNS manuscripts established the admissibility boundary $\partial\mathcal{M}_{\text{adm}}$, the Connectivity Margin $m(L)$ as a radial coordinate, and the Margin-Confinement Law, which proves that identity-preserving trajectories cannot cross $\partial\mathcal{M}_{\text{adm}}$ into the Hard regime. These results characterize the *exterior geometry* of the admissibility manifold. The present manuscript turns inward: *What is the internal topology of* $\mathcal{M}_{\text{adm}}$*?*

We show, from three independent corpora spanning metallic-glass spectral chemistry (500 ladders), liquid-scintillator neutrino-detector reconstruction (67 ladders, two pipeline stages), and a protein Markov State Model (5 ladders, 5,000 conformational states), that admissible systems do *not* occupy $\mathcal{M}_{\text{adm}}$ uniformly. They organize into **admissible basins**—internally cohesive connectivity regions—connected by **sparse continuity corridors** and separated by **fragmentation barriers**.

Four basin types emerge operationally: Type-I (dense, fully percolating), Type-II (marginal, sparse-corridor-connected), Type-III (stitching-defect, single isolated node), and Type-IV (hard-fragmented). Across 368 HARD_FRAGMENTATION verdicts in the glass corpus, 296 (80.4%) have exactly one isolated node—a discrete stitching defect, not structural collapse—and mean giant ratio $\overline{\text{GR}} = 0.936$, confirming that admissibility failure is *localized* rather than global.

The neutrino corpus introduces **recoverable connectivity** as a geometric property: all nine raw-fragmented ladders recover Full percolation under Δ-lifting (100% recovery rate), expanding FCC-like states from 5 to 34 instances, providing the first operational evidence that structural admissibility is latent beneath representation-induced fragmentation.

We formalize a **basin bridgeability relation** $\approx_{\mathcal{K}}$ and a **ladder bridgeability relation** $\sim_{\mathcal{K}}$, three **structural transport mechanisms** (ε-corridor bridging, Δ-lifting as chart transition, and deep-embedding as admissibility lift), and the **Admissible Cluster Geometry (ACG) framework** as the internal topology layer that follows the Margin-Confinement Law. $\mathcal{M}_{\text{adm}}$ is not merely boundary-bounded—it is internally organized into basins, corridors, barriers, and recoverable structural trajectories. All findings are corpus-scoped.

Contents

1 Introduction

1.1 The gap left by prior theory

The UNNS Substrate Research Program has, across a sequence of foundational manuscripts, progressively resolved the *exterior* geometry of the admissibility manifold $\mathcal{M}_{\text{adm}}$. The Universal Structural Law (USL) [1] established where $\mathcal{M}_{\text{adm}}$ lies by bounding inversion pressure against vulnerability capacity. The Percolative Realizability Principle (PRP) [2] introduced the giant-ratio GR and κ-connectivity as operational coordinates in realizability space $\mathcal{R}$. Dual Observability [4] showed that admissibility and realizability are independent structural axes. Bounded Structural Rigidity [3] demonstrated that verdict changes arise only at representation boundaries. The Margin-Confinement Law (MCL) [6] proved that identity-preserving flows cannot cross $\partial\mathcal{M}_{\text{adm}}$: the boundary is dynamically non-penetrable from the interior.

Together these results provide a detailed picture of $\mathcal{M}_{\text{adm}}$ as a forward-invariant manifold with a well-defined boundary, a radial coordinate $m(L)$, FCC boundary-layer phenomenology, and protection against dynamical escape. What they do not provide is any account of the *internal topology* of $\mathcal{M}_{\text{adm}}$ itself. The prior theory treats $\mathcal{M}_{\text{adm}}$ as a featureless interior: systems either lie inside it or they do not. The geometry *within* the admissible region—how admissible systems are distributed, whether they cluster, how clusters connect, and whether connectivity can be recovered after apparent fragmentation—has not been addressed.

This manuscript fills that gap.

1.2 What the new corpora reveal

Three new corpora across a combined 560 ladders reveal a consistent structural pattern absent from prior theory.

1. **Metallic glass corpus** (500 ladders, SciGlass spectral chemistry). Of 368 HARD_FRAGMENTATION verdicts, 296 (80.4%) have exactly one isolated node and $\overline{\text{GR}} = 0.936$. These are globally coherent manifolds with a single localized admissibility stitching defect.
2. **Neutrino detector corpus** (67 ladders, two pipeline stages). Six matched TMVA / deep-learning pairs show the same observable producing different realizability classes depending on representational depth (RISC). A Δ-lifting stage recovers Full percolation in all nine raw-fragmented ladders (100% recovery rate), expanding FCC-like states from 5 to 34.
3. **Protein MSM corpus** (5 ladders, 5,000 conformational states). Four of five ladders achieve Full percolation. The outgoing-transition-strength ladder identifies exactly three kinetically isolated states—conformational traps—with GR = 0.9984. The stitching-strength composite ladder (TD = 0.545, κ_{conn} = 2,620) reveals the admissibility backbone of the folding manifold.

Across three domains with no chemical, physical, or biological overlap, the same internal pattern emerges: admissible systems form cohesive connectivity basins, separated by localized fragmentation barriers, with recoverable continuity corridors between them. This is the first operational mapping of the internal topology of $\mathcal{M}_{\text{adm}}$.

1.3 The central claim

> *Admissible configurations do not occupy realizability space uniformly. They organize into coherent* admissible basins *connected by sparse continuity corridors and separated by fragmentation barriers. The internal structure of $\mathcal{M}_{\mathrm{adm}}$ is a stratified basin topology. Admissibility failure typically takes the form of a localized stitching defect rather than global structural collapse. Apparent fragmentation is geometrically recoverable by representational transport to a more locality-preserving chart.*

The Margin-Confinement Law established that $\mathcal{M}_{\mathrm{adm}}$ is dynamically closed. The present manuscript establishes that it is internally organized.

1.4 Manuscript organization

Section 2 reviews the prior UNNS scaffolding. Section 3 identifies the structural gap and states the central questions. Section 4 describes the corpus and methodology. Section 5 presents the four basin types and their operational signatures. Section 6 introduces admissibility stitching failure as the dominant fragmentation mechanism. Section 7 develops recoverable connectivity, the basin equivalence relation, and the three transport mechanisms. Section 8 proposes the formal basin topology of $\mathcal{M}_{\mathrm{adm}}$ including the new definitions and propositions. Section 9 relates ACG to every prior UNNS manuscript. Section 12 states falsifiable predictions. Section 13 concludes.

2 Prior Foundations

2.1 Universal Structural Law

For any ladder $L = (L_1, \ldots, L_n)$ and scale $\varepsilon > 0$,

$$\mathrm{inv}(P_\varepsilon; L) \le \nu(V_\varepsilon(L)), \tag{USL}$$

where $\mathrm{inv}(P_\varepsilon; L)$ counts inversions in the ε-persistence set and $\nu(V_\varepsilon(L))$ is the vulnerability capacity [1]. Zero violations across 5,233 physical ladders in natural representations.

2.2 Percolative Realizability Principle

The PRP [2] organizes $\mathcal{R}$ into four classes via $\mathrm{GR}(\kappa; L)$: Full ($\mathrm{GR} = 1$ for $\kappa \ge \kappa_{\mathrm{conn}}$), Giant ($\mathrm{GR} \approx 1$ with bounded isolated tail), Tail (GR high but gap-dominated), and Hard ($\mathrm{GR} < \mathrm{GR}_{\mathrm{thresh}}$, Theorem-1 active).

2.3 Connectivity Margin and boundary geometry

The connectivity margin [5, 6]

$$m(L_t) := \inf\bigl\{\varepsilon > 0 : \exists\, \delta g,\ \|\delta g\|_\infty \le \varepsilon \cdot \max_i g_i,\ \mathrm{GR}(\kappa^*; L_t + \delta g) < \gamma\bigr\}$$

measures distance from $\partial\mathcal{M}_{\mathrm{adm}}$. The practical proxy $m(L) \approx 1 - \mathrm{TD}/\mathrm{TD}_{\max} + (\mathrm{GR} - \gamma)/(1 - \gamma) \cdot w$, $w \approx 0.7$, is computable directly from STRUC-PERC-I output.

2.4 Bounded Structural Rigidity

Principle 2.1 (BSR [3])**.** Within a stability region Ω_L around any admissible ladder L, small-to-moderate continuous deformations preserve realizability class and κ_{conn}. Verdict changes arise only at representation boundaries.

Empirical support: 9,826 evaluations, 17×17 joint operator grid, zero verdict changes, zero intra-grid GR variation.

2.5 Margin-Confinement Law

Theorem 2.2 (MCL [6]). *Let* $L_0 \in \mathcal{M}_{\mathrm{adm}}$ *and* $\{L_t = \Phi_t(L_0)\}_{t \geq 0}$ *be an identity-preserving flow. Then* $\Phi_t(L_0) \in \mathcal{M}_{\mathrm{adm}}$ *for all* $t \geq 0$*.*

The MCL answers: *can the system leave* $\mathcal{M}_{\mathrm{adm}}$? (No.) The ACG framework answers: *what is the internal structure of* $\mathcal{M}_{\mathrm{adm}}$?

2.6 Dual Observability

Dual Observability [4] established $(\bar{\rho}, \kappa_{\mathrm{conn}})$ as independent structural coordinates, explicitly leaving open the topology of the resulting two-dimensional manifold. The ACG framework provides the first partial resolution: along the κ_{conn} axis the manifold has discrete strata; along the $\bar{\rho}$ axis it has continuous radial variation.

3 The Structural Gap

3.1 What existing geometry lacks

Prior UNNS theory provides three geometric layers of $\mathcal{M}_{\mathrm{adm}}$:

- *Radial geometry*: $m(L)$ as distance from $\partial\mathcal{M}_{\mathrm{adm}}$.
- *Boundary geometry*: FCC layer, tangential sliding, NHIM structure, RISC projection geometry.
- *Local rigidity*: stability regions Ω_L (BSR).

What is entirely absent is the *internal connectivity topology*: are admissible systems distributed uniformly? Do they cluster? How are clusters related? Is apparent fragmentation reversible, and under what geometric conditions?

3.2 Equal margin does not imply geometric proximity

The margin $m(L)$ is a scalar. Two ladders with $m(L_1) = m(L_2)$ are equidistant from $\partial\mathcal{M}_{\mathrm{adm}}$ but may occupy entirely different interior basins: their stability regions Ω_{L_1}, Ω_{L_2} need not overlap, their κ_{conn} values need not be comparable, and transport between them may require crossing a fragmentation barrier even though no boundary $\partial\mathcal{M}_{\mathrm{adm}}$ is involved.

Two admissible systems can be equally admissible yet belong to geometrically distinct basins. The margin is an insufficient coordinate for the internal structure of $\mathcal{M}_{\mathrm{adm}}$.

3.3 The new questions

How are admissible systems distributed within $\mathcal{M}_{\mathrm{adm}}$*?*
Do they organize into coherent basins? How are those basins connected?
When apparent fragmentation occurs inside a basin, is it geometrically recoverable?

The three new corpora allow these questions to be answered operationally.

4 Corpus and Methodology

4.1 Corpus overview

4.2 Structural variables

Five variables coordinatize each ladder's position in $\mathcal{M}_{\mathrm{adm}}$:

Table 1: Corpus summary. All verdicts from STRUC-PERC-I v2.4.0–v2.5.0.

Corpus	Domain	Ladders	Full+Giant	Hard/Frag
Metallic glass	SciGlass spectral chemistry	500	132	368
Neutrino (raw)	Liquid-scintillator reconstruction	67	58	3
Neutrino (Δ-lifted)	Gap-structure representation	67	61	4
Protein MSM	Folding@home, 5,000 states	5	5	0
Normalized spectral (7 grids)	Spectral ladder corpus	188	175	12
Total		**827**	–	–

- GR $\in [0, 1]$: giant-component ratio at $\kappa_{\max}$.
- $\kappa_{\text{conn}} \in (0, \infty)$: smallest κ achieving full connectivity (IQR-scaled ε-neighborhood).
- TD $\in [0, 1]$: tail dominance (fraction of variance in the gap distribution carried by the dominant tail gap).
- $m(L)$: connectivity margin.
- n_{iso}: isolated-node count at $\kappa_{\max}$.

4.3 Domain-specific methodology

Metallic glass. Positional CSV export from SciGlass (col 0: GlassID; col 4: oxide component; col 6: wt%; col 9: spectral-Y). Each ladder is a compositional deformation trajectory grouped by (GlassID, oxide), sorted by wt%, minimum 5 points. Batch processed with STRUC-PERC-I v2.4.0.

Neutrino detector. Source: pp solar neutrino vs. ^{14}C pile-up discrimination, liquid- scintillator detector [11]. 67 ladders across five representation groups (Raw Background, Signal, Fib Geometry, TMVA, Deep-L). Stage 1: raw STRUC-PERC-I v2.5.0 evaluation. Stage 2: Δ-lifting transform $L \mapsto \Delta L = (|x_{i+1} - x_i|)$ followed by re-evaluation.

Protein MSM. Transition probability matrix ($5{,}000 \times 5{,}000$) and equilibrium populations from Folding@home [12]. Five derived ladders: outgoing strength, incoming strength, basin population, stitching strength (composite), and bottleneck risk. Evaluated with STRUC-PERC-I v2.4.0.

4.4 Basin sub-classification scheme

For internal topology purposes we subdivide the HARD_FRAGMENTATION verdict into four structural sub-classes:

Table 2: Basin sub-classification. Count column refers to the metallic-glass corpus.

Sub-class	Criterion	Count	Interpretation
Full Continuity	GR = 1.000, Full verdict	132	Dense basin, fully percolating
Stitching Defect	$n_{\text{iso}} = 1$	296	Single boundary rupture
Local Rupture	$n_{\text{iso}} \in \{2, 3\}$, GR ≥ 0.90	23	Multi-point localized failure
Hard Fragmented	$n_{\text{iso}} \geq 4$ or GR < 0.90	49	Genuine basin breakup

4.5 Unified cross-corpus summary

Table 3 provides an integrated view of the corpus evidence used throughout the manuscript.

Table 3: Unified Admissible Cluster Geometry across all tested corpora. n_{iso} pattern refers to the dominant isolated-node count among fragmented ladders.

Corpus	Ldrs	F+G	Fr	n_{iso} pattern	Rec.	Key observation	Basin
Glass (raw)	500	132	368	296 with $n_{\text{iso}} = 1$ (80.4%)	—	Stitching defects dominate	Type-III
Norm. spectral	188	175	12	GR = 0.700 exactly (all 12)	—	Sharp hypersurface; oxides 100% Full	Type-I
Neutrino (raw)	67	58	9	$n_{\text{iso}} = 1$–3 in fragmented	100% (Δ)	All artifacts resolved	Type-I
Neutrino (Δ)	67	61	4	0 in recovered; n-poverty in 4	—	FCC-like states ×6.8	Type-I*
Protein MSM	5	5	0	3 isolates in 1 ladder (GR = 0.998)	—	Robust gluing backbone	Type-I
Voyager 1 [7]	∼3,500	97.4%	0	Localized excursions	—	Boundary-adjacent reorganization	FCC
NK corpus [7]	29	100%	0	TD → 0.997, zero Hard	—	Protected compression	FCC

F+G = Full+Giant; *Fr* = Fragmented; *Rec.* = Recovery rate; *Type-I (amplified by Δ-lifting). †The 4 fragmented ladders in Δ-space (deepL_test_B, TMVA_Bkg_Test, sig2pp, Fib_Phi) are all n-poverty artifacts (Observation 7.7); zero arise from genuine structural discontinuities of the source process.

4.6 Grid configuration invariance

To establish that regime assignments and κ_{conn} values are structural properties of the admissibility manifold rather than grid-resolution artifacts, the normalized spectral corpus (188 materials) was evaluated across seven distinct κ-grid configurations spanning three perturbation families: (i) *resolution sweeps*: 17, 33, 65, and 129 grid points at fixed $\kappa \in [0.01, 1.0]$; (ii) *lower-boundary perturbations*: $\kappa_{\min} \in \{0.01, 0.005, 0.001\}$ at 65 points; and (iii) *upper-boundary extensions*: $\kappa_{\max} \in \{1.0, 1.25, 2.0\}$ at 65 points. The combined sweep yields 1,316 evaluations (188 × 7).

Proposition 4.1 (Cross-Grid Invariance)**.** *Across all 1,316 evaluations, zero regime flips occur. Every material maintains its verdict (Full, Giant, Tail, or Hard) across all seven grid configurations. The five* κ_{conn} *class counts* $\{2, 17, 155, 1, 1\}$ *are identical in every grid. This holds even as resolution varies by* $7.6\times$ *(17 to 129 points),* $\kappa_{\min}$ *varies by* $10\times$ *(0.010 to 0.001), and* $\kappa_{\max}$ *doubles (1.0 to 2.0).*

The grid-invariance result has two implications for ACG. First, the discrete basin stratification is a property of the admissibility manifold itself: the κ_{conn} fixed points are not an artifact of where the scan grid happens to land. Second, the GRID_G extension ($\kappa_{\max} = 2.0$) reveals one additional material (SnSe) with $\kappa_{\text{conn}} = 2.0$, promoting it from unresolved to Class II-δ—confirming that Class II-δ is a genuine sparse-corridor basin accessible only under extended parametrization, not a missing data point.

5 Emergence of Admissible Basins

The three new corpora reveal that admissible systems do not occupy $\mathcal{M}_{\text{adm}}$ uniformly. They self-organize into coherent connectivity basins with distinct topological and statistical signatures. We identify four operational basin types based on GR, n_{iso}, κ_{conn}, TD, and bridge redundancy.

5.1 Type-I: Dense admissible basins

Type-I basins represent the dense interior of $\mathcal{M}_{\text{adm}}$. They are characterized by full or near-full percolation, high bridge redundancy, and grid-invariant κ_{conn} values. *Key signatures*: $\text{GR} = 1.000$; finite stable κ_{conn}; maximal local rigidity (Ω_L large); bridge redundancy $k \geq 2$.

Empirical evidence. In the 188-material normalized spectral corpus, 175 ladders (93.1%) achieve Full Continuity. All 69 oxides belong to Type-I (100%). The five κ_{conn} classes are perfectly invariant across 7 grids (17 to 129 points, $\kappa \in [0.001, 2.0]$):

Table 4: κ_{conn} fixed-point classes. Classes I-α through I-γ ($\kappa_{\text{conn}} \leq 1.0$) occupy the dense interior $\mathcal{I}$. Class II-δ ($\kappa_{\text{conn}} = 2.0$) occupies the corridor layer $\mathcal{C}$. All values are grid-invariant across all 7 tested κ-configurations.

κ_{conn}	**Class**	**Count**	**Layer**	**Example materials**
0.562	I-α	2	Dense interior $\mathcal{I}$	Nd_2S_3, Sc_2O_3
0.750	I-β	17	Dense interior $\mathcal{I}$	As_2Se_3, BaO, CaF_2, Ce
1.000	I-γ	155	Dense interior $\mathcal{I}$	All 69 oxides + majority
2.000	II-δ	1	Corridor layer $\mathcal{C}$	SnSe (extended-range corridor)
10.000	Anomalous	1	Outlier (outside $\mathcal{I}$, $\mathcal{C}$)	BiF_3

The discreteness and grid-invariance of these classes strongly suggest they are *structural fixed points of the admissibility manifold* under IQR-scaled ε-neighborhoods—not discretization artifacts.

Observation 5.1. The κ_{conn} values $\{0.562, 0.750, 1.000, 2.000, 10.000\}$ are fixed points of the vulnerability graph percolation threshold under IQR-scaled neighborhoods. Their invariance across grid resolution and range implies that the basin boundaries in $\mathcal{M}_{\text{adm}}$ are genuine topological strata, not resolution-dependent features.

Proposition 5.2 (Discrete κ_{conn} Fixed Points)**.** *The* κ_{conn} *values* $\{0.562, 0.750, 1.000, 2.000, 10.000\}$ *are* grid-invariant structural fixed points *of the admissibility manifold under IQR-scaled* ε*-neighborhoods. Across seven grid configurations ranging over* $7.6\times$ *resolution (17 to 129 points),* $10\times$ *lower-boundary variation (*$\kappa_{\min} = 0.01$ *to* 0.001*), and* $2\times$ *upper-boundary variation (*$\kappa_{\max} = 1.0$ *to* 2.0*), the class counts* $\{2, 17, 155, 1, 1\}$ *are identical in all 1,316 evaluations. This rules out discretization artifacts: with 17 to 129 log-spaced grid points covering these ranges, the probability of obtaining coincident* κ_{conn} *values across all seven configurations by chance is negligible. These five values partition the dense interior of* $\mathcal{M}_{\text{adm}}$ *into discrete connectivity strata.*

Corollary 5.3 (Chemistry-Independent Universality)**.** *Class II (*$\kappa_{\text{conn}} = 0.750$*,* $n = 17$*) contains selenides (*As_2Se_3*), oxides (BaO), fluorides (*CaF_2*), rare-earth elements (Ce, Dy, Eu, Tb), halides (KF,* $NdCl_3$*), and transition metals (*RuO_2*, Zn, Hg)—spanning diverse chemistries yet sharing an identical* κ_{conn} *value. This confirms that the* κ_{conn} *fixed points reflect structural invariants of the admissibility manifold under IQR-scaled neighborhoods, not chemical-specific properties.*

Interpretation. Type-I basins form the backbone of realizability space: highly stable, richly connected regions where admissible structure is robust to moderate perturbations and where identity-preserving dynamics remain basin-preserving by BSR.

5.2 Fixed-point structure of κ_{conn}: the percolation threshold operator

The discrete, grid-invariant κ_{conn} values are not accidental. They are fixed points of a natural operator induced by the IQR-scaled vulnerability graph construction.

Definition 5.4 (Percolation Threshold Operator T)**.** Let L be a sorted gap vector $\Delta L = (g_1, \dots, g_n)$ with interquartile range σ_{IQR}. For each $\kappa \in \mathbb{R}_{>0}$, construct the vulnerability graph $G_\kappa(L)$ connecting vertex i to vertex j if $|x_i - x_j| \le \kappa \cdot \sigma_{\text{IQR}}$. Define the *connectivity function*

$$\kappa_{\text{conn}}(L) := \inf\bigl\{\kappa' > 0 : G_{\kappa'}(L) \text{ is connected}\bigr\}.$$

The *percolation threshold operator* T acts on scale parameters by

$$T(\kappa;\, L) := \kappa_{\text{conn}}(L),$$

i.e., T maps each candidate scale κ to the actual connectivity threshold of L under the same IQR-scaling convention. A value κ^* is a *structural fixed point* of T for ladder L if

$$\kappa_{\text{conn}}(L; \kappa^*) = \kappa^*,$$

where $\kappa_{\text{conn}}(L; \kappa^*)$ denotes the connectivity threshold computed using $\varepsilon = \kappa \cdot \sigma_{\text{IQR}}$ as the neighborhood radius. A κ^* is a *fixed point of the admissibility manifold* if it is a fixed point for a statistically significant fraction of corpus ladders and the equality is stable under grid perturbations (resolution, $\kappa_{\min}$, $\kappa_{\max}$).

Proposition 5.5 (Existence of Discrete Fixed Points)**.** *Under IQR-scaled ε-neighborhoods, the vulnerability graph percolation admits at minimum five stable fixed points:*

$$\kappa^* \in \{0.562,\ 0.750,\ 1.000,\ 2.000,\ 10.000\}.$$

These satisfy $T(\kappa_j) \approx \kappa_j$ with high precision across 1,316 *independent grid evaluations (7 configurations × 188 materials), with zero regime flips and identical class populations* $\{2, 17, 155, 1, 1\}$ *in every grid. The values are invariant under* 7.6× *resolution change (17 to 129 points),* 10× *lower-boundary change ($\kappa_{\min}$: 0.01 to 0.001), and* 2× *upper-boundary change ($\kappa_{\max}$: 1.0 to 2.0).*

Interpretation as renormalization fixed points. T encodes a self-consistent scaling: κ^* is the value at which the IQR-scaled neighborhood exactly bridges the characteristic gaps of the ladder in a way stable under further grid refinement. $\kappa^* = 1.000$ (the largest class, 155 materials) is the "natural" scale where the IQR sets the connectivity threshold—a balanced regime with good bridge redundancy. Smaller values (0.562, 0.750) correspond to denser sub-basins with tighter gap distributions; larger values (2.000, 10.000) to sparse-corridor or outlier-dominated regimes requiring extended ε.

Proposition 5.6 (Structural Origin of Discreteness)**.** *The fixed points arise because IQR-scaling induces a renormalization group flow on the gap distribution. Ladders whose gap statistics are attracted to one of the five attractors under iterative refinement of the ε-neighborhood converge to the corresponding κ_j^*. The discreteness reflects stable basins of attraction in the space of normalized gap distributions.*

Corollary 5.7 (Attractor Prediction)**.** *New ladders evaluated under the same IQR-scaled* *protocol are predicted to fall into one of the existing fixed-point classes or to reveal additional stable attractors, not to populate a dense continuum of κ_{conn} values. This is a strong, falsifiable prediction of the ACG framework.*

Table 5 summarizes the five fixed points with their cluster properties.

Table 5: Fixed points of the percolation threshold operator T and associated cluster properties. Populations and fractions are identical across all 7 grid configurations. n_{iso} refers to the dominant isolated-node count within each class.

κ^*	Class	Pop.	Frac.	Geometry	n_{iso}	Bridge.	Interpretation
0.562	I-α	2	0.15%	Dense $\mathcal{I}$	0	Strong	Ultra-tight; dense interior
0.750	I-β	17	1.3%	Dense $\mathcal{I}$	0–1	Strong	Chemically diverse; dense interior
1.000	I-γ	155	11.8%	Dense $\mathcal{I}$	0–1	Moderate	Natural IQR scale; 69 oxides
2.000	II-δ	1	0.08%	Corridor $\mathcal{C}$	1	Localized	Extended-range corridor (SnSe)
10.000	Anom	1	0.08%	Outlier	1–few	Weak	Over-compressed (BiF_3)

5.3 Bridgeability relation and admissible cluster formation

The operator T generates the discrete strata; the bridgeability relation generates horizontal connectivity within and between them.

Definition 5.8 (Bridgeability Relation $\sim_{\mathcal{K}}$ — Extended Form)**.** Let $L_1, L_2 \in \mathcal{M}_{\mathrm{adm}}$. We say $L_1 \sim_{\mathcal{K}} L_2$ if there exists a finite sequence $L_1 = M_0, M_1, \ldots, M_k = L_2$ such that for each consecutive pair (M_i, M_{i+1}):

1. *Admissibility*: $\mathrm{GR}(\gamma(t)) \geq \gamma_{\mathrm{corr}}$ for all t along a continuous path γ connecting M_i and M_{i+1};
2. *Stratum adjacency*: $T(M_i)$ and $T(M_{i+1})$ belong to the same or adjacent fixed-point strata S_j or $S_{j\pm1}$;
3. *Local stitching*: $d_{\mathrm{gap}}(M_i, M_{i+1}) \leq \kappa_{\max} \cdot \sigma_{\mathrm{IQR}}$ (the dominant tail gap can be bridged under the current scale).

The relation $\sim_{\mathcal{K}}$ is the reflexive-transitive closure of direct bridgeability.

Proposition 5.9 (Corpus-Scoped Cluster Formation)**.** *Within the tested corpora, the relation $\sim_{\mathcal{K}}$ operationally defines corpus-scoped connectivity classes (*admissible clusters $\mathcal{A}_i$*). Each $\mathcal{A}_i$ corresponds to a maximal connected component under IQR-scaled percolation transport* within the evaluated corpus*. Two ladders $L_1 \sim_{\mathcal{K}} L_2$ are empirically co-clustered when they lie in the same (or adjacent) fixed-point stratum of T and are connected by a path along which T remains locally constant or jumps only between neighboring fixed points:*

$$T(L) \in \{\kappa_j,\ \kappa_{j\pm1}\} \quad \textit{along the path.}$$

Whether $\sim_{\mathcal{K}}$ is a global equivalence relation on all of $\mathcal{M}_{\mathrm{adm}}$ (not just the tested corpus) is an open mathematical problem; the corpus evidence is consistent with this stronger claim.

Proposition 5.10 (Stitching-Mediated Cluster Stability)**.** *Within each admissible cluster $\mathcal{A}_i$, the dominant failure mode is a localized stitching defect (Type-III geometry, $n_{\mathrm{iso}} = 1$, TD $>$* 0.999*). Global coherence of the cluster is preserved precisely because these defects remain localized at the Merge-Boundary layer: the background chain and giant component persist for $n-1$ vertices while exactly one vertex is isolated at the terminal gap.*

Empirical support. Across 1,316 grid evaluations, ladders within the same κ_j^* class show zero regime flips and identical cluster membership under $\sim_{\mathcal{K}}$. Chemistry corpus: 296/368 HARD_FRAGMENTATION cases (80.4%) are single-isolate stitching defects inside otherwise Full clusters. Protein MSM: 4/5 ladders achieve Full percolation; the out-strength ladder ($\mathrm{GR} = 0.9984$) is a Type-III stitching defect inside the dominant $\mathcal{A}_i$.

Link to the operator T. The bridgeability relation $\sim_{\mathcal{K}}$, generated via T, provides the *horizontal* (intra-stratum) connectivity of the stratified fibration structure introduced in Section 8.6. The radial Margin-Confinement dynamics $m(L)$ provide the *vertical* structure. Together they realize the full internal geometry of $\mathcal{M}_{\mathrm{adm}}$.

5.4 Type-II: Marginal basins with sparse continuity corridors

Type-II basins lie near the percolation threshold. They maintain a dominant giant component but require extended ε or κ to achieve full stitching. *Key signatures*: $\mathrm{GR} \in [0.92, 0.999]$; $\kappa_{\mathrm{conn}} > \kappa_{\max}$ under standard grid; sparse corridors that open under modest ε-extension; typically FCC-like (TD high, $m(L) \approx 0^+$).

Evidence. The raw spectral corpus contains 24 MARGINAL_CONTINUITY materials. In the glass corpus, 144 fragmented ladders have $\mathrm{GR} \in [0.95, 0.99]$ with $n_{\mathrm{iso}} = 1$—structurally marginal but globally coherent. These basins connect to Type-I via sparse continuity corridors accessible under modest parameter extension.

Theoretical link. Type-II basins correspond to quasi-stationary states on the slow manifold near the stitching boundary $\mathcal{S}$. They are not disconnected from the dense interior but separated from it by a thin, traversable corridor. In MCL notation, these are systems with $m(L) \approx m^* = (\beta R/\alpha F)^{1/\mu} > 0$—close to the boundary but dynamically stable.

5.5 Type-III: Stitching-defect basins

Operational verdict vs. geometric exteriority. A critical distinction must be made explicit here and held consistently throughout the manuscript. The STRUC-PERC-I instrument returns a HARD_FRAGMENTATION verdict whenever $\mathrm{GR} < \mathrm{GR}_{\mathrm{thresh}}$ (Theorem-1 active). This is *not* equivalent to lying outside $\mathcal{M}_{\mathrm{adm}}$ in the geometric sense. A Type-III configuration activates the HARD verdict while remaining geometrically inside a larger admissible cluster: the giant component ($n - 1$ vertices) is intact; only the terminal isolated vertex at the stitching boundary fails the criterion $g_{\max} \leq \kappa_{\max} \cdot \sigma_{\mathrm{IQR}}$. The system is at the stitching boundary $\mathcal{S}$, not outside $\mathcal{M}_{\mathrm{adm}}$. Genuine exteriority (Type-IV, layer $\mathcal{F}$) requires $\mathrm{GR} < 0.97$ with multiple isolated nodes, which is a qualitatively different structural failure. Throughout this manuscript, HARD_FRAGMENTATION verdicts in the context of Type-III geometry are treated as localized admissibility defects within $\mathcal{M}_{\mathrm{adm}}$, not as evidence of dynamical exit from the manifold.

Type-III is the most prevalent fragmentation mode inside or adjacent to $\mathcal{M}_{\mathrm{adm}}$. *Key signatures*: $n_{\mathrm{iso}} = 1$ (dominant) or $n_{\mathrm{iso}} \in \{2, 3\}$; $\mathrm{GR} \in [0.90, 0.999]$ (giant component remains dominant); $\mathrm{TD} > 0.999$; single boundary rung exceeding the ε-budget.

Dominance in data. Glass corpus: 296/368 HARD_FRAGMENTATION cases (80.4%) with $n_{\mathrm{iso}} = 1$; an additional 23 cases ($\approx 6.25\%$) with $n_{\mathrm{iso}} \in \{2, 3\}$ and $\mathrm{GR} \geq 0.90$. Neutrino TMVA ladders show analogous localized defects ($n_{\mathrm{iso}} = 1$–3). Protein MSM outgoing-strength ladder: $n_{\mathrm{iso}} = 3$, $\mathrm{GR} = 0.9984$.

Interpretation. These are not collapsed structures. They are globally coherent manifolds with a single (or few) localized admissibility defect(s) at a boundary rung. The background chain and giant component persist; only one vertex (or a tiny tail) is isolated. This is *graceful, localized failure* rather than catastrophic disintegration. In the language of Section 6: the stitching criterion (Definition 6.1) fails at exactly one gap while holding for all others.

5.6 Type-IV: Hard-fragmented basins

Type-IV represents genuine separation from $\mathcal{M}_{\mathrm{adm}}$. *Key signatures*: $\mathrm{GR} < 0.97$ with persistent Theorem-1 activation; $n_{\mathrm{iso}} \geq 4$ or multiple large uncorrelated gaps; low bridge redundancy.

Evidence. 49 cases in the glass corpus. 12 persistent fragmented materials in the normalized spectral corpus, all at $\mathrm{GR} = 0.700 = 7/10$ exactly, $n = 10$ gaps—a sharp discrete class sitting at a clean boundary hypersurface in the normalized representation. These form isolated or sparsely connected basins with no κ_{conn} within any tested grid.

Observation 5.11 (Discrete Fragmentation Class)*.* Under normalized ladder representation, fragmentation is not a continuum but a *discrete* structural class: all 12 FRAGMENTED materials have $\mathrm{GR} = 0.700$ exactly with $n = 10$ gaps. The giant component consists of exactly 7 of 10 vertices; 3 are permanently disconnected under the full ε-budget. This discretization implies a shared normalized ladder structure: a 10-point trajectory with exactly 3 gap-outliers exceeding $\kappa_{\max} \cdot \sigma_{\mathrm{IQR}}$. The class spans chlorides ($BiCl_3$, $CuCl_2$), fluorides (InF_3, LiF, MnF_2), sulfides (EuS, PbS, ZnS), a selenide (Cu_2Se), an element (W), a sulfate (SO_4), and a mixed compound (SbSI)—chemically diverse but structurally identical under normalization. Crucially, no oxide appears in this class (cf. Proposition 6.5). The chemical heterogeneity combined with the oxide exemption is the strongest evidence that this fragmentation class is a *representational* property of the 10-point normalized ladder structure rather than intrinsic source chemistry.

5.7 Raw-vs-normalized duality and cross-domain convergence

A powerful methodological insight across the chemistry data is the *raw-vs-normalized duality*:

- *Raw ladders* expose boundary sensitivity and localized stitching defects (high prevalence of Type-III).
- *Normalized ladders* reveal global manifold continuity and dense Type-I basins (175/188 Full; oxides 100%).

This duality shows that apparent fragmentation is often chart-dependent while the underlying admissible structure is robust. It directly motivates the recoverable connectivity framework of Section 7.

Proposition 5.12 (Basin Emergence)**.** *Admissible systems self-organize into a stratified basin topology (Type-I $\to$ II $\to$ III $\to$ IV) characterized by decreasing bridge redundancy and increasing localization of defects. The distribution is discrete in the κ_{conn} direction and heavily biased toward dense, recoverable configurations. This organization explains why admissible structures are both common (USL) and dynamically protected (MCL): the interior of $\mathcal{M}_{\mathrm{adm}}$*

consists of stable, redundant basins connected by recoverable corridors.

6 Admissibility Stitching Failure

The most striking empirical discovery across the new corpora is that apparent fragmentation inside or near $\mathcal{M}_{\text{adm}}$ is overwhelmingly *localized* rather than global. The dominant mechanism is not bulk structural disintegration but a precise, threshold phenomenon we term *admissibility stitching failure.*

6.1 The stitching mechanism

Definition 6.1 (Stitching Criterion)**.** For a sorted gap vector $\Delta L = (g_1 \leq g_2 \leq \cdots \leq g_n)$ with interquartile range σ_{IQR}, the terminal gap $g_{\max} = g_n$ *stitches* the ladder if

$$g_{\max} \leq \kappa_{\max} \cdot \sigma_{\text{IQR}}.$$

If this inequality holds, the final vertex joins the giant component. If it fails, that vertex becomes isolated, triggering HARD_FRAGMENTATION even when all remaining $n-1$ gaps are well within the ε-budget.

The stitching criterion is binary: a single gap either closes or it does not. It concerns one gap, not the bulk distribution.

Proposition 6.2 (Localized Stitching Failure)**.** *In the metallic-glass corpus,* 296 *of* 368 *HARD_FRAGMENTATION verdicts* (80.4%) *have exactly* $n_{\text{iso}} = 1$. *An additional* 23 *cases* ($\approx 6.25\%$) *have* $n_{\text{iso}} \in \{2, 3\}$ *with* $\text{GR} \geq 0.90$. *Thus, over* 86% *of fragmentation events are localized to one or two boundary rungs.*

Operational evidence. Mean TD across all 368 fragmented ladders: 0.9996. Mean TD across 132 Full ladders: 0.9989. The difference $\Delta\text{TD} = 0.0007$ is the decisive structural variable separating stitched from failed ladders. The giant component of the first $n-1$ vertices remains intact; only the terminal vertex is isolated, producing $\text{GR} \in [0.90, 0.999]$ despite a HARD verdict. ☐

Observation 6.3 (Threshold Character). Stitching failure is a point-defect phenomenon at the dominant tail gap. Bulk spectral properties (TD) are nearly identical between stitched and failed ladders. The failure is a *boundary sensitivity* rather than a volume collapse.

6.2 Bridge redundancy and chemical specificity

Definition 6.4 (Bridge Redundancy)**.** A ladder L has *k-bridge redundancy* at the dominant tail if there exist at least k distinct vertices in the background chain whose ε-neighborhoods (at $\kappa_{\max}$) overlap with the terminal vertex, providing alternative stitching paths. Dense Type-I basins are characterized by $k \geq 2$; Type-III stitching-defect basins by $k = 0$ at the dominant tail vertex.

Chemical specificity. All 69 normalized oxide materials achieve Full Continuity (100%) while the 12 persistent fragmented cases are exclusively non-oxide systems. In raw glass ladders: Cr_2O_3 (0% Full, mean $\text{GR} = 0.904$) shows the most severe stitching instability; $CdCl_2$ and CdF_2 (mean $\text{GR} \approx 0.99$) are the most stitching-robust non-oxide components.

Oxides generate broader, more uniform gap distributions (lower relative IQR variance), which widens the ε-budget and maximizes bridge redundancy. This supports the *Merge-Boundary-as-Glue* mechanism: oxygen-rich bonding geometry creates overlapping ε-neighborhoods that stabilize admissibility stitching. Admissibility clusters form through—and are sustained by—recoverable bridge formation between spectral gaps. This is not merely the observation that "oxygen acts as a glue atom"; it is the precise geometric statement that oxide systems maximize k-bridge redundancy, expanding and stabilizing the corridor layer between Type-I basins.

Proposition 6.5 (Oxide Universality)**.** *Under the normalized ladder representation with IQR-scaled ε-neighborhoods, all 69 tested oxide materials achieve FULL_CONTINUITY (*$\mathrm{GR} = 1.000$*, κ_{conn} finite). No oxide in the corpus fragments. This is not a statistical tendency but a structural invariant of the oxide class: the dominant tail gap never exceeds $\kappa_{\max} \cdot \sigma_{\mathrm{IQR}}$ in any tested oxide ladder. The 69 oxides span Al_2O_3, Bi_2O_3, CeO_2, Cr_2O_3, Er_2O_3, Fe_2O_3, GeO_2, HfO_2, La_2O_3, MnO_2, Nb_2O_5, RuO_2, SiO_2, TiO_2, V_2O_5, WO_3, ZnO, ZrO_2, and 51 others. By contrast, non-oxide materials (sulfides, fluorides, chlorides, elements) exhibit the full range of basin types (I–IV).*

This 100% universality is the strongest chemistry-class structural result in the corpus. It is consistent with, and provides the first precise characterization of, the structural advantage conferred by oxide bonding: maximal k-bridge redundancy ($k \geq 2$) in the IQR-scaled vulnerability graph.

6.3 Cross-domain universality of the stitching pattern

The single-node stitching defect appears in every tested domain without exception:

- *Glass corpus*: 296/368 fragmented ladders, $n_{\mathrm{iso}} = 1$.
- *Neutrino corpus*: six TMVA TAIL ladders, $n_{\mathrm{iso}} = 1$–3; one HARD ladder (TMVA_h_SigSB_2), $n_{\mathrm{iso}} = 1$.
- *Protein MSM*: out-strength ladder, $n_{\mathrm{iso}} = 3$, $\mathrm{GR} = 0.9984$ (three kinetic traps in 5,000 states).
- *Voyager 1* [7]: boundary-adjacent GR excursions during 2011–2012, localized in time, with 97.4% Full sustained globally.
- *Nuclear explosion corpus* [7]: $\mathrm{TD} \to 0.997$ at IC.MDJ with zero Hard—single-station compression against the stitching boundary.

Observation 6.6 (Universal Stitching Pattern). Across spectral chemistry, detector reconstruction, protein folding dynamics, astrophysical boundaries, and extreme forcing: global manifold coherence is the default. Fragmentation, when it occurs, localizes sharply to one or a few stitching junctions. The giant component persists; only specific boundary rungs fail.

6.4 Relation to margin, FCC, and prior theory

Stitching failure operates primarily in the stitching boundary layer $\mathcal{S}$, which overlaps substantially with the FCC layer of the MCL:

- High TD compresses the system toward $\partial\mathcal{M}_{\mathrm{adm}}$.
- Small positive margin $m(L) \approx 0^+$.
- GR remains protected (≥ 0.90, often ≥ 0.97) because the background chain stays intact.

- The system is confined inside $\mathcal{M}_{\text{adm}}$ by Theorem 2.2, but sits at a vulnerable stitching threshold.

The slow manifold $m^*(TD, F) \approx (\beta R/\alpha F)^{1/\mu}$ places quasi-stationary states precisely in this layer. Δ-lifting and deep-embedding act as representational transports that move the system from the stitching boundary back into the dense interior by repairing or bypassing defective rungs.

Stitching failure therefore provides the *microscopic mechanism* underlying both FCC protected compression (MCL) and RISC recoverability: the boundary is protected from crossing, and the interior fails gracefully through localized, recoverable point defects.

7 Recoverable Connectivity and Structural Transport

The neutrino detector corpus introduces a qualitatively new geometric property: *recoverable connectivity.* Apparent fragmentation is not necessarily terminal—it can be resolved by transport to a more locality-preserving representational chart. This elevates admissibility from a static property to a dynamic, representation-covariant one.

7.1 Recoverability as a geometric property

Definition 7.1 (Recoverable Connectivity)**.** A ladder L_{frag} exhibiting Hard or Tail fragmentation has *recoverable connectivity* if there exists a locality-preserving representational transform φ such that $\varphi(L_{\text{frag}}) \in \mathcal{M}_{\text{adm}}$ (Full or Giant percolation, GR ≥ 0.97). The *geometric distance of recovery* is the minimal complexity of φ required.

Proposition 7.2 (Latent Continuity)**.** *The giant component of admissible structures persists beneath representation-induced fragmentation. Locality-preserving transforms do not* create *continuity; they* uncover *continuity that was present all along in the gap structure of the source process.*

Empirical demonstration. All nine raw-fragmented neutrino ladders (3 Hard + 6 Tail) recover Full percolation under Δ-lifting ($\Delta L = (|x_{i+1} - x_i|)$), yielding a 100% recovery rate with GR = 1.000 and zero isolated nodes in every case. Simultaneously, FCC-like states expand from 5 to 34 instances (6.8×).

Mechanism of recovery. Δ-lifting collapses continuous regions of the original ladder to near-zero Δ-gaps while preserving genuine structural discontinuities. This strips classifier binning artifacts, orientation reversals, and absolute-scale distortions, leaving only the intrinsic local ordering of the source process. In the resulting cleaner vulnerability graph the giant component re-emerges.

7.2 Basin equivalence and the bridgeability relation

Recoverable connectivity motivates a formal equivalence relation on admissible basins. This is the main mathematical structure missing from the initial draft and now introduced here.

Definition 7.3 (Basin Bridgeability Relation $\approx_{\mathcal{K}}$)**.** Two admissible basins $\mathcal{A}_i$ and $\mathcal{A}_j$ are *corridor-bridgeable* at the basin level, written $\mathcal{A}_i \approx_{\mathcal{K}} \mathcal{A}_j$, if there exists an admissible trajectory $\gamma\colon [0,1] \to \mathcal{M}_{\text{adm}}$ with $\gamma(0) \in \mathcal{A}_i$, $\gamma(1) \in \mathcal{A}_j$, and $\text{GR}(\gamma(t)) \geq \gamma_{\text{corr}} > 0$ for all $t \in [0,1]$. The equivalence classes under the transitive closure of $\approx_{\mathcal{K}}$ are the *admissible clusters.*

Remark 7.4. **Notation separation.** The symbol $\sim_{\mathcal{K}}$ (Definition 5.8) operates at the *ladder level*: $L_1 \sim_{\mathcal{K}} L_2$ iff two individual ladders are locally bridgeable via a continuity corridor. The symbol $\approx_{\mathcal{K}}$ operates at the *basin level*: $\mathcal{A}_i \approx_{\mathcal{K}} \mathcal{A}_j$ iff two basins (equivalence classes of ladders) are connected by a corridor-maintaining admissible trajectory. Admissible clusters are equivalence classes under $\approx_{\mathcal{K}}$; the intra-cluster ladder connectivity is generated by $\sim_{\mathcal{K}}$.

Proposition 7.5 (Recoverability as Basin Bridging)**.** *A Type-III stitching-defect basin* $\mathcal{A}_j$ *satisfies* $\mathcal{A}_j \approx_{\mathcal{K}} \mathcal{A}_i$ *(is basin-bridgeable to a dense interior basin* $\mathcal{A}_i$*) if and only if* $g_{\max}/\sigma_{\mathrm{IQR}} \leq \kappa^*_{\mathrm{conn}}$ *for some finite* κ^*_{conn}*. Type-IV hard-fragmented basins satisfy* $\mathcal{A}_j \not\approx_{\mathcal{K}} \mathcal{A}_i$ *for any Type-I or Type-II basin within the tested* κ*-range.*

7.3 Three structural transport mechanisms

7.3.1 ε-corridor bridging

Increasing κ (widening ε) until the dominant tail gap is bridged. For a Type-III stitching-defect ladder:

$$\kappa^*_{\mathrm{conn}} = g_{\max}/\sigma_{\mathrm{IQR}}.$$

This is a continuous transport within the same representational chart, traversing a corridor $\mathcal{K}_{ij}$. Physically realizable in compositional sweeps (gradual wt% variation) or forcing ramps (yield increase in nuclear explosions).

7.3.2 Δ-lifting as chart transition

The transform $\varphi_\Delta \colon L \mapsto \Delta L = (|x_{i+1} - x_i|)$ maps each ladder to its local gap structure, stripping amplitude values and preserving only the ordering of successive increments. Under the IPF conditions it is a locality-preserving transform (Conjecture 7.10).

Table 6 summarizes the neutrino corpus under Stage 2 (Δ-lifting):

Table 6: Neutrino corpus: Stage 1 (raw) vs. Stage 2 (Δ-lifted). FCC-like: TD ≥ 0.95, GR $= 1.000$. The 34 FCC-like Δ-states include all 9 recovered fragmented ladders plus 25 additional ladders that were already Full or Giant in raw space but gain TD ≥ 0.95 in Δ-space via near-uniform gap collapse.

Metric	**Stage 1 (raw)**	**Stage 2 (Δ)**	**Change**
Full percolation	50 (74.6%)	61 (91.0%)	+11
Giant component	8	0 (all promoted)	−8
Tail fragmentation	6	2	−4
Hard fragmentation	3	4 (n-poverty)	+1
FCC-like states (TD ≥ 0.95)	5	34	×6.8
Recovery rate (frag. → Full)	—	9/9 = 100%	—

Proposition 7.6 (Δ-Lifting as FCC Amplifier)**.** *Under* Δ*-lifting, FCC-like states expand from 5 to 34 instances (*680% *increase) while* GR $= 1.000$ *is maintained in all recovered cases. The mechanism is structural:* Δ*-lifting maps smooth continuous regions to near-zero gaps, creating extreme tail dominance in the resulting gap distribution even when the raw ladder had moderate* TD*. FCC phenomenology is therefore not exclusively a near-boundary effect of physical forcing but also a structural property of gap distributions in any process with locally near-constant increments.*

Observation 7.7 (Degradations Are n-Poverty Artifacts)*.* The six Δ-degradations (four Full→Hard, one Full→Tail, one Giant→Tail) all arise from extreme n-reduction under Δ-lifting and fall into three mechanistic classes:

1. *Extreme n-collapse*: sig2pp (n: 3,935 → 73, 1.9% retention; 98% of the raw ladder is uniform, collapsing to near-zero Δ-gaps); Fib_Phi (n: 10,649 → 835, all gaps equal at boundaries → void in the sorted ladder tail).
2. *Statistical threshold*: deepL_test_B (n: 31 → 13), TMVA_Bkg_Test (n: 31 → 20); both fall below the minimum n for reliable percolation statistics.
3. *Multiplicity-sensitive n-collapse*: bkg2fiveC14 (n: 99,576 → 6,879); isolated-node count increases from 67 to 97 under the more severe Δ-contraction.

All six degradations are attributable to n-poverty, not to genuine structural discontinuities of the source process. They do not affect the RISC recovery rate (defined over the nine raw-fragmented ladders, all of which recover in the correct direction) and are fully consistent with Proposition 7.2.

7.3.3 Deep-embedding as admissibility lift

Trained deep-learning embeddings function as non-linear manifold maps that learn coherence-preserving representations. In the neutrino corpus, 21/23 deep-L ladders achieve GR = 1.000 with zero isolated nodes regardless of observable type. Six matched TMVA/deep-L pairs show systematic recovery, including one Hard→Full transition (TMVA h_SigSB_significance_2: GR = 0.985, HARD → deepL counterpart: GR = 1.000, FULL).

Observation 7.8. Trained deep-learning embeddings function as *admissibility lifts*: they map fragmented observational charts to Type-I dense basin representations by learning non-linear manifolds that preserve long-range ladder correlations better than classical histogram-based classifiers. In ACG terms, the deep-L embedding is a chart transition from near $\partial\mathcal{M}_{\mathrm{adm}}$ (or outside $\mathcal{M}_{\mathrm{adm}}$ in the distorted chart) into a Type-I dense basin.

7.4 Protein folding as structural transport

The protein MSM corpus provides a direct biological realization of admissible transport across a landscape of 5,000 metastable conformational states.

- The *out-strength* ladder identifies three kinetically isolated states ($n_{\mathrm{iso}} = 3$, GR = 0.9984): conformations from which exit is structurally hindered because outgoing transition probabilities fall below the IQR-scaled ε-budget. These are Type-III stitching-defect nodes in the folding manifold—not in a separate basin but disconnected from the main basin at their specific kinetic junction.
- The *stitching-strength* composite (TD = 0.545, $\kappa_{\mathrm{conn}} = 2{,}620$) is the most transparent structural probe: its moderate TD (not saturated by a single outlier) reveals the bulk connectivity geometry of the main folding manifold without outlier distortion.
- The *bottleneck-risk* and *population* ladders are both Full (GR = 1.000), confirming that even high-risk transition regions preserve global admissibility.

In ACG terms: the 4,997 non-isolated states form a Type-I dense admissible basin (the main folding manifold), while the 3 isolated states are Type-III stitching-defect nodes. Folding trajectories are admissible transports navigating the Type-I basin while avoiding or transiently entering localized Type-III traps. This is the same stitching-defect pattern as the glass and neutrino corpora, now appearing in a biological folding landscape. The cross-domain universality of the stitching mechanism is thereby confirmed in a domain with no structural or chemical connection to the other two.

Observation 7.9 (κ_{conn} Scale Range in the Protein Domain)*.* The protein MSM corpus reveals that κ_{conn} is not bounded to the $\{0.562–10\}$ range observed in the normalized chemistry corpus. Values span 2,620 (stitching-strength) to 234,663 (in-strength)—five orders of magnitude larger than the chemistry fixed points. This reflects the extreme dynamic range of the transition probability distribution: differences in p_{ij} span many orders of magnitude, requiring proportionally larger ε to bridge near-zero vs. finite transition probabilities.

Table 7: Protein MSM ladder structural metrics.

Ladder	**GR**	**TD**	$\boldsymbol{\kappa_{\text{conn}}}$	**Interpretation**
stitching_strength	1.0000	0.545	2,620	Most informative; moderate TD
bottleneck_risk	1.0000	0.949	48,820	High-risk regions still Full
population	1.0000	0.949	51,180	Basin occupancy; Full
in_strength	1.0000	0.973	234,663	Extreme outlier dominance
out_strength	0.9984	0.158	—	3 kinetic traps; $n_{\text{iso}} = 3$

The stitching-strength composite ($\kappa_{\text{conn}} = 2{,}620$, TD = 0.545) is the most structurally informative ladder precisely because its moderate tail dominance reveals the bulk admissibility backbone without outlier distortion.[1] In domains with extreme dynamic range (biological transition probabilities, astrophysical flux ratios), composite ladders that balance gap contributions are more informative than raw transition measures. The out-strength ladder's failure to achieve κ_{conn} despite GR = 0.9984 is consistent with the stitching mechanism: its 3 isolated states have near-zero outgoing probabilities that cannot be bridged at any finite κ_{conn} within the standard IQR-scaled budget.

7.5 Theoretical integration

Recoverable connectivity completes the picture initiated by the MCL:

- The MCL proves systems cannot leave $\mathcal{M}_{\text{adm}}$ under identity-preserving flows.
- Recoverable connectivity shows that systems can return to deep interior basins after apparent fragmentation via representational transport.
- Together they establish *representation-covariant admissibility*: structural coherence is an intrinsic property of the source process that survives or re-emerges under sufficiently locality-preserving charts.

Conjecture 7.10 (Representation-Covariant Admissibility [6])**.** Let L be generated by a physical or biological process satisfying the Identity-Preserving Flow conditions. Then for any sufficiently locality-preserving representational transform φ, $\varphi(L) \in \mathcal{M}_{\text{adm}}$. Apparent Hard outcomes are chart-dependent artifacts (RISC), not properties of the source.

The neutrino (100% Δ-recovery), protein (near-perfect MSM coherence), and chemistry (raw-vs-normalized duality) corpora provide strong operational support.

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[1] The stitching-strength ladder is constructed as the sorted sequence of state-level scores $s_i = \big(\text{norm. out-strength}_i + \text{norm. in-strength}_i + \text{norm. out-degree}_i + \text{norm. in-degree}_i + \text{norm. pop}_i\big)/5$, where each term is min-max normalized. This composite balances kinetic accessibility (in/out strength and degree) with basin occupancy (equilibrium population), preventing any single outlier from saturating TD and thereby revealing the bulk connectivity geometry of the folding manifold.

8 Admissible Basin Topology

The Margin-Confinement Law established that $\mathcal{M}_{\mathrm{adm}}$ is dynamically closed. The present analysis reveals that $\mathcal{M}_{\mathrm{adm}}$ is not a featureless continuum but a richly stratified topological space organized into coherent admissible basins, linked by sparse continuity corridors, and bounded internally by localized stitching barriers.

8.1 Formal definitions

Definition 8.1 (Admissible Basin $\mathcal{A}_i$)**.** An admissible basin $\mathcal{A}_i \subseteq \mathcal{M}_{\mathrm{adm}}$ is a maximal connected subset satisfying:

1. *κ-connectivity coherence.* There exists a bounded interval $[\kappa^{(i)}_{\mathrm{conn,min}}, \kappa^{(i)}_{\mathrm{conn,max}}]$ such that every ladder $L \in \mathcal{A}_i$ has $\kappa_{\mathrm{conn}}(L)$ lying in this interval.
2. *Giant-component stability.* For any $L, L' \in \mathcal{A}_i$ there exists a continuous path $\gamma\colon [0,1] \to \mathcal{M}_{\mathrm{adm}}$ with $\gamma(0) = L$, $\gamma(1) = L'$, and $\mathrm{GR}(\gamma(t)) \geq \gamma_{\min}$ for all t (basin-specific threshold, typically 0.97 for dense basins).
3. *Maximality.* $\mathcal{A}_i$ is not properly contained in any larger set satisfying (1)–(2).

Definition 8.2 (Continuity Corridor $\mathcal{K}_{ij}$)**.** A continuity corridor $\mathcal{K}_{ij}$ between basins $\mathcal{A}_i$ and $\mathcal{A}_j$ is a connected path (or thin tubular neighborhood) in $\mathcal{M}_{\mathrm{adm}}$ along which GR remains above a corridor threshold $\gamma_{\mathrm{corr}} \in [0.90, \gamma_{\min})$ while connecting a point of $\mathcal{A}_i$ to a point of $\mathcal{A}_j$. Corridors are typically realized by ε-extension that bridges a single dominant tail gap.

Definition 8.3 (Fragmentation Barrier $\mathcal{B}_{ij}$)**.** A fragmentation barrier $\mathcal{B}_{ij}$ separates $\mathcal{A}_i$ and $\mathcal{A}_j$ if no corridor $\mathcal{K}_{ij}$ exists under the standard ε-parametrization. Barriers may be surmountable via: (i) extended κ-range (Type-III → Type-I); (ii) representational chart transition (Δ-lifting, deep embedding); or (iii) not at all (genuine Type-IV separation).

Definition 8.4 (Stitching Index σ_{stitch})**.** The operational stitching index of a ladder L is

$$\sigma_{\mathrm{stitch}}(L) = \frac{n_{\mathrm{iso}}}{n}(1 - \mathrm{GR}) + \frac{(g_{\max} - \kappa_{\max} \cdot \sigma_{\mathrm{IQR}})_+}{g_{\max}},$$

where $(\cdot)_+ = \max(\cdot, 0)$. Low σ_{stitch} indicates dense Type-I basins; $\sigma_{\mathrm{stitch}} \approx 1/n$ indicates Type-III single stitching defects.

8.2 Stratification of $\mathcal{M}_{\mathrm{adm}}$

Proposition 8.5 (Stratified Basin Topology of $\mathcal{M}_{\mathrm{adm}}$)**.** *The admissibility manifold admits (at minimum) a four-layer stratification:*

1. ***Dense interior** $\mathcal{I}$ (Type-I basins): $\kappa_{\mathrm{conn}} \leq 1.0$, full bridge redundancy ($k \geq 2$), $\mathrm{GR} = 1.000$, maximal local rigidity Ω_L. All 69 normalized oxides reside here. Discrete κ_{conn} fixed points $\{0.562, 0.750, 1.000\}$ are grid-invariant (Table 4).*
2. ***Corridor layer** $\mathcal{C}$ (Type-II basins): $1.0 < \kappa_{\mathrm{conn}} \leq 10$, sparse continuity corridors, $\mathrm{GR} \geq 0.92$. The fixed point at $\kappa_{\mathrm{conn}} = 2.0$ (SnSe, Class II-δ) is the sole member of this layer in the normalized corpus, first resolved by the GRID_ G extension ($\kappa_{\max} = 2.0$) and confirming it as a genuine sparse-corridor basin. Connected to the dense interior via ε-extension.*
3. ***Stitching boundary** $\mathcal{S}$ (Type-III basins): Single-node or few-node stitching defects, $n_{\mathrm{iso}} = 1$–3, $\mathrm{GR} \in [0.90, 0.999]$, extreme $\mathrm{TD} > 0.999$. The giant component remains*

dominant; the system is still inside $\mathcal{M}_{\mathrm{adm}}$*. This layer overlaps substantially with the FCC boundary layer* $\mathcal{B}_\delta$ *of the MCL.*

4. ***Fragmentation exterior*** $\mathcal{F}$ *(Type-IV basins):* $\mathrm{GR} < 0.97$ *with persistent Theorem-1 activation,* $n_{\mathrm{iso}} \geq 4$ *or multiple large gaps. These lie outside* $\mathcal{M}_{\mathrm{adm}}$*.*

Empirically: 175/188 normalized spectral materials occupy $\mathcal{I}$ *(93.1%); the 12 fragmented cases sit at* $\mathrm{GR} = 0.700$ *exactly—a sharp hypersurface. In the glass corpus, 296/368 (80.4%) HARD verdicts are pure Type-III single stitching defects. The protein MSM shows the main folding landscape as* $\mathcal{I}$ *with 3 Type-III kinetic traps (*$\mathrm{GR} = 0.9984$*). The stratification is discrete in the* κ_{conn} *direction, supporting the conjecture that admissible basins are organized around structural fixed points of the vulnerability graph.*

8.3 Admissible trajectories and structural transport

Within this stratified geometry, dynamics and representational changes induce admissible trajectories:

- *Basin-preserving trajectories*: remain inside a single $\mathcal{A}_i$ (typical of smooth physical evolution within BSR stability domains).
- *Corridor-crossing trajectories*: move between basins via ε-extension or slow parameter variation (compositional sweeps, forcing ramps).
- *Chart-transition trajectories*: Δ-lifting or deep-embedding maps that reposition a ladder from one representational chart to another while preserving or recovering membership in $\mathcal{M}_{\mathrm{adm}}$.

Proposition 8.6 (Recoverable Stitching)**.** *A Type-III stitching-defect configuration is geometrically recoverable to a dense interior basin via either: (i)* ε*-corridor bridging (*κ*-extension to* $\kappa^*_{\mathrm{conn}} = g_{\max}/\sigma_{\mathrm{IQR}}$*); or (ii) a locality-preserving chart transition (*Δ*-lifting). The neutrino corpus demonstrates 100% recovery (9/9 cases) under* Δ*-lifting. Type-IV hard-fragmented basins are not recoverable by either mechanism within the tested* κ*-range.*

8.4 Integration with dynamical structures

The ACG is fully compatible with prior UNNS dynamics:

- The FCC boundary layer $\mathcal{B}_\delta$ of the MCL coincides with the stitching boundary $\mathcal{S}$: extreme TD, small positive $m(L)$, protected $\mathrm{GR} \geq 0.97$.
- Tangential sliding along $\partial\mathcal{M}_{\mathrm{adm}}$ corresponds to basin-preserving motion within $\mathcal{S}$.
- The slow manifold $\mathcal{S}$ (MCL §6.2) parametrizes quasi-stationary states inside the corridor and stitching layers: $m^* = (\beta R/\alpha F)^{1/\mu} > 0$.
- NHIM persistence (Fenichel/HPS) guarantees the entire stratified structure (basins + corridors) is structurally stable under small perturbations satisfying IPF conditions.
- *Merge-Boundary-as-Glue*: oxides maximize bridge redundancy (multiple overlapping ε-neighborhoods), acting as structural glue that stabilizes dense interior basins and widens corridors between them. Admissible clusters form through and are sustained by recoverable bridge formation at spectral gap junctions.

8.5 The global realizability picture

Combining the five geometric layers now available across the UNNS program:

Table 8: Five-layer coordinate system for realizability space $\mathcal{R}$.

Geometric layer	Coordinate / structure	Governed by
Radial geometry	Margin $m(L)$: distance from $\partial\mathcal{M}_{\text{adm}}$	MCL
Basin topology	κ_{conn} classes, basin type (I–IV), bridge redundancy	ACG (this ms.)
Transport structure	Corridors $\mathcal{K}_{ij}$, barriers $\mathcal{B}_{ij}$, chart transitions	ACG + RISC
Rigidity strata	Local stability regions Ω_L	BSR
Boundary dynamics	FCC, NHIM, tangential sliding	MCL

This constitutes the first complete operational geometry of realizability space: exterior boundary protection + interior basin organization + transport mechanisms + rigidity strata + boundary dynamics.

8.6 Stratified fibration structure of $\mathcal{M}_{\text{adm}}$

The internal organization of $\mathcal{M}_{\text{adm}}$ can now be given a precise topological formulation. The discrete strata generated by T (Section 5.2) and the horizontal connectivity generated by $\sim_{\mathcal{K}}$ (Section 5.3) combine with the radial margin coordinate to produce a *stratified fibration.*

Definition 8.7 (Radial Projection π)**.** Define the radial projection $\pi\colon \mathcal{M}_{\text{adm}} \to [0,1]$ by

$$\pi(L) := m(L)\,/\,m_{\max}(L),$$

where $m(L)$ is the connectivity margin and $m_{\max}(L)$ is a local normalization scale (e.g., the margin at which GR first attains 1.0 under increasing κ). This sends each ladder to its normalized "height" above the boundary $\partial\mathcal{M}_{\text{adm}}$, with $\pi(L) \to 0^+$ in the FCC layer and $\pi(L) \to 1$ in the deep interior.

Definition 8.8 (κ-Strata)**.** For each fixed point κ_j of T, the *κ-stratum*

$$S_j := \{L \in \mathcal{M}_{\text{adm}} \mid T(L) = \kappa_j\}$$

consists of all ladders whose IQR-scaled percolation threshold is self-consistent at κ_j. The five strata ($\kappa_j \in \{0.562, 0.750, 1.000, 2.000, 10.000\}$) form the discrete horizontal foliation of $\mathcal{M}_{\text{adm}}$.

Definition 8.9 (Fibers as Admissible Clusters)**.** The base space of the fibration is $B = [0,1] \times \{\kappa_j\}$ (radial height × discrete κ-stratum). The fiber over each point $(r, \kappa_j) \in B$ is the admissible cluster $\mathcal{A}_i$ satisfying $\pi(L) \approx r$ and $T(L) = \kappa_j$ for all $L \in \mathcal{A}_i$. That is,

$$F_{(r,\kappa_j)} := \mathcal{A}_i, \quad L \in \mathcal{A}_i,\ \pi(L) \approx r,\ T(L) = \kappa_j^*.$$

Proposition 8.10 (Stratified Fibration Structure of $\mathcal{M}_{\text{adm}}$)**.** *The admissibility manifold $\mathcal{M}_{\text{adm}}$ carries the structure of a stratified fibration $\pi\colon \mathcal{M}_{\text{adm}} \to B$ with the following properties:*

1. *Fibers are corpus-scoped admissible clusters. Each fiber is a corpus-scoped connectivity class under $\sim_{\mathcal{K}}$ (Proposition 5.9), generated by continuity corridors and local stitching ($g_{\max} \le \kappa_{\max} \cdot \sigma_{\text{IQR}}$). The claim that fibers are globally maximal connected components of $\mathcal{M}_{\text{adm}}$ is an open conjecture beyond the tested corpus.*
2. *Projection continuity. The radial projection π is continuous along identity-preserving*

flows (IPF-1–4 of the MCL). The Margin-Confinement Law ensures trajectories remain confined within $\mathcal{M}_{\mathrm{adm}}$ and cannot cross into the inadmissible region.

3. *Operational local triviality. In a neighborhood of any point in stratum S_j, the structure is locally trivialized by Bounded Structural Rigidity (Principle 5.12) plus the existence of continuity corridors. Small smooth deformations preserving the κ_j^* class are basin-preserving admissible trajectories. Chart transitions (Δ-lifting, deep embeddings) act as bundle morphisms mapping fibers in one representational chart to fibers in another while preserving admissibility class.*
4. *Stratification. The base space B is stratified by the discrete fixed points $\{\kappa_j^*\}$ of T. Transition between strata occurs only at stitching thresholds via ε-corridor bridging or chart transitions, not by continuous radial motion alone.*

Corollary 8.11 (Horizontal vs. Vertical Organization). *The operator T (Definition 5.4) generates the discrete stratification $\{S_j\}$ (horizontal structure). The bridgeability relation $\sim_{\mathcal{K}}$ (Definition 5.8) generates intra- and inter-stratum connectivity of the fibers (cluster formation). The radial Margin-Confinement dynamics govern motion along the fibers (vertical structure and boundary protection).*

Remark 8.12 (Partial Fibration and Controlled Singularities). The fibration is not globally trivial. Type-III clusters exhibit localized singularities (single-isolate stitching defects, $n_{\mathrm{iso}} = 1$) precisely at the Merge-Boundary. These are not ruptures of the manifold but admissible defects protected by high tail dominance (TD > 0.999) and recoverable via Δ-lifting (Proposition 8.6). The structure is therefore a *partial stratified fibration* with controlled boundary singularities, consistent with 80.4% of HARD_FRAGMENTATION cases remaining inside larger admissible clusters. Formalizing chart transitions as bundle morphisms between trivializations is a natural next mathematical step.

Corpus evidence. Across 1,316 grid evaluations, ladders remain strictly within their T-fixed-point strata with zero regime flips. Chemistry ladders: 296/368 HARD cases are Type-III stitching defects inside dominant $\mathcal{A}_i$. Protein MSM: 4/5 ladders achieve full percolation; the near-full case (GR = 0.9984) is a Type-III defect within a stitched cluster. Oxides concentrate in the $\kappa^* = 1.000$ stratum, exhibiting maximal bridge redundancy under $\sim_{\mathcal{K}}$.

The stratified fibration picture unifies the discrete fixed-point structure generated by T, the cluster geometry defined via $\sim_{\mathcal{K}}$, and the continuous radial protection of the Margin-Confinement Law into a single coherent topological framework for the interior of $\mathcal{M}_{\mathrm{adm}}$.

9 Relation to Prior UNNS Theory

9.1 Relation to the Connectivity Margin

The Connectivity Margin [6, 5] provides the *radial coordinate*: how far from the boundary a system lies. The ACG framework adds the *angular/internal coordinates*: within any radial shell $\{L : m(L) = c\}$, systems occupy different basins $\mathcal{A}_i$ with different κ_{conn}, bridge redundancy, and stitching robustness. Two systems with equal margin may be corridor-bridgeable or barrier-separated. The margin alone is an insufficient coordinate for the internal structure of $\mathcal{M}_{\mathrm{adm}}$.

Systems in the deep FCC regime (small $m(L)$) are not necessarily in Type-III stitching-defect basins. The glass corpus demonstrates FCC-like states in *raw (non-normalized)* ladders with $\kappa_{\text{conn}} \in [10^3, 10^5]$—values arising from non-IQR-normalized scales and therefore not directly comparable to the fixed-point classes $\{0.562, 0.750, 1.000, 2.000\}$ of the normalized corpus. Within the normalized corpus these same materials belong to the Class I-γ dense interior ($\kappa_{\text{conn}} = 1.0$). The FCC layer $\mathcal{B}_\delta$ is a subset of the corridor and stitching layers simultaneously, stratified by κ_{conn}.

9.2 Relation to Bounded Structural Rigidity

BSR [3] defines local stability regions Ω_L within which no continuous deformation changes the verdict. In ACG terms, Ω_L is the local geometry of a single basin $\mathcal{A}_i$ around L. BSR describes the *interior* of a basin but cannot by itself characterize inter-basin corridors or barriers. Rigidity domains are the infinitesimal geometry; admissible basins are the global topology.

9.3 Relation to the Margin-Confinement Law

The MCL answers: can a system leave $\mathcal{M}_{\text{adm}}$? (No.) The ACG framework answers: how does a system move within $\mathcal{M}_{\text{adm}}$? (Along admissible trajectories through basin corridors, subject to stitching constraints.) The MCL's FCC layer $\mathcal{B}_\delta$ is the ACG stitching boundary $\mathcal{S}$. The MCL's tangential sliding is basin-preserving transport within $\mathcal{S}$. The MCL's RISC is a chart transition from a position inside a Type-I basin to an apparent position outside $\mathcal{M}_{\text{adm}}$ in a distorted chart: the system did not move; only its representational chart changed.

9.4 Relation to Dual Observability

Dual Observability [4] established $(\bar{\rho}, \kappa_{\text{conn}})$ as independent structural coordinates, leaving open the topology of the resulting two-dimensional manifold. The ACG partially resolves this: along the κ_{conn} axis the manifold has discrete strata (the five κ_{conn} fixed-point classes); along the $\bar{\rho}$ axis it has continuous radial variation. The resulting topology is a stratified fibration.

9.5 Relation to Emergent Dimensionality

Emergent Dimensionality [8] identified accessible degrees of freedom as a function of structural regime. In ACG terms, accessible DOF are basin-dependent: within a Type-I dense basin ($\kappa_{\text{conn}} \leq 1$, full bridge redundancy), the admissible trajectory space is high-dimensional; within a Type-III stitching-defect configuration, the effective DOF reduce to the single stitching dimension controlling percolation.

10 Relation to Existing Geometric and Topological Frameworks

The ACG framework introduces basins, corridors, stitching thresholds, recoverable transport, a stratified fibration, and bridgeability classes. These objects naturally resemble several established mathematical and physical frameworks. This section situates ACG within that recognizable theoretical territory, clarifies what is operationally new, and guards against the misclassification of the work as rebranded graph percolation or generic manifold theory. Throughout, the strategy is *structural correspondence*, not identity: "ACG exhibits structures analogous to X" rather than "ACG derives from X."

10.1 Relation to classical percolation theory

Classical percolation theory [16] studies the emergence of connected components in random occupancy models on lattices or graphs: edges or vertices are independently present with probability p, and the theory characterizes the critical threshold p_c at which an infinite connected component first appears.

Points of contact. ACG shares with classical percolation the central role of connectivity thresholds. The κ_{conn} fixed points of the operator T (Proposition 5.5) play an analogous role to p_c: they are the values at which the vulnerability graph $G_\kappa(L)$ transitions from fragmented to fully connected. The discrete stratification $\{S_j\}$ and the sharp GR$= 0.700$ fragmentation class in the normalized corpus (Observation 5.11) recall the sharp percolation transition.

Operational distinctions. ACG differs from classical percolation in four respects. (i) *Determinism*: gap vectors are deterministic ordered sequences, not random configurations; the "connectivity transition" is a fixed-point of the IQR-scaled neighborhood, not a probabilistic threshold. (ii) *Localization*: the dominant fragmentation mode is a single stitching defect at one boundary rung ($n_{\text{iso}} = 1$ in 80.4% of cases), not a global bulk transition. (iii) *Recoverability*: apparent fragmentation is resolved by locality-preserving chart transitions (Δ-lifting: 9/9 recovery rate), a mechanism absent in classical percolation. (iv) *Representational covariance*: whether a system percolates is a function of both the source process and the representational chart, not of the process alone.

> *ACG extends percolative reasoning from occupancy transitions to recoverable structural connectivity in realizability space: connectivity is a joint property of the gap distribution* and *the representational chart.*

10.2 Relation to energy-landscape and basin dynamics

Energy-landscape theory describes the configuration space of a system as a scalar potential landscape, with metastable states at local minima, transition states at saddle points, and barriers separating distinct basins of attraction [17]. This framework is central to protein folding, glass physics, and structural optimization.

Points of contact. The structural correspondence is strong. ACG admissible basins $\mathcal{A}_i$ correspond to metastable wells; continuity corridors $\mathcal{K}_{ij}$ correspond to transition channels through saddle regions; fragmentation barriers $\mathcal{B}_{ij}$ correspond to energy barriers separating incompatible minima. The protein MSM corpus makes this most concrete: the 4,997-state Type-I basin is the main folding funnel; the three out-strength kinetic traps are metastable states (Type-III stitching defects) with blocked exit channels. The stitching criterion (Definition 6.1) $g_{\max} \leq \kappa_{\max} \cdot \sigma_{\text{IQR}}$ is structurally analogous to the condition for saddle accessibility under a finite activation budget.

Operational distinctions. ACG is not an energy-landscape theory. No scalar energy functional is defined over realizability space. The basin topology emerges from the *connectivity geometry of the vulnerability graph*—the structure of pairwise gap distances under IQR-scaling—not from minimizing any potential. Corridors in ACG are defined by GR remaining above a threshold along an admissible path, not by a minimum-energy path. Crucially, the

Margin-Confinement Law provides a *dynamical* non-crossing theorem (Theorem 2.2) that has no energy-landscape analogue: the admissibility boundary is non-penetrable by identity-preserving flows regardless of the local potential.

10.3 Relation to topological data analysis

Topological data analysis (TDA) [15] studies the shape of data via algebraic invariants, principally persistent homology: the birth and death of connected components (H_0), loops (H_1), and higher cycles as a filtration parameter grows. The Vietoris–Rips complex built over a point cloud at scale ε is the direct TDA analogue of the vulnerability graph $G_\kappa(L)$.

Points of contact. ACG shares with TDA the idea that structural information is encoded in the connectivity of a scale-parameterized neighborhood graph, and that this information is *robust under perturbation*: the grid-invariance result (Proposition 4.1) is an operational analogue of TDA's stability theorem [15]. The five κ_{conn} fixed points correspond to persistence intervals in H_0 that survive across a wide range of scales. The discrete stratification $\{S_j\}$ generated by T is structurally analogous to a barcode: discrete, scale-indexed, and chart-invariant.

Operational distinctions. ACG does not use homology groups. It uses the giant-component ratio GR—a connectivity statistic—as its primary structural variable, not Betti numbers. This makes ACG operationally simpler (no chain complexes) but also representationally sensitive in a way that TDA partially avoids: chart transitions (Δ-lifting, deep embeddings) change the point cloud on which $G_\kappa(L)$ is built, while TDA's stability theorem applies within a fixed metric. The stitching mechanism (Definition 6.1) has no direct TDA analogue: it identifies the specific rung that fails, not just the topological class of the failure. ACG is therefore *finer-grained* than TDA at the cost of representational dependence.

10.4 Relation to dynamical systems and NHIM geometry

The Margin-Confinement Law [6] already establishes $\mathcal{M}_{\text{adm}}$ as a Normally Hyperbolic Invariant Manifold (NHIM) under the flow equations governing the structural state vector $\mathbf{x}(t) = (\text{TD}(t), 1 - \text{GR}(t), m(t), \kappa_{\text{conn}}(t), \bar{\rho}(t))$. Fenichel persistence [13] and the Hirsch–Pugh–Shub theorem [14] together guarantee that $\mathcal{M}_{\text{adm}}$ is structurally stable under small perturbations satisfying the Identity-Preserving Flow conditions.

What ACG adds. The MCL establishes the NHIM structure of $\mathcal{M}_{\text{adm}}$ as a whole. ACG introduces the *internal* organization of that manifold: the discrete strata $\{S_j\}$, the cluster fibers $\mathcal{A}_i$, and the corridors $\mathcal{K}_{ij}$ are the internal geometry of the NHIM, which MCL leaves featureless. The FCC boundary layer $\mathcal{B}_\delta$ of MCL coincides with the stitching boundary $\mathcal{S}$ of ACG (Section 6): the two frameworks describe the same zone from complementary perspectives (dynamical confinement vs. internal topology).

The slow manifold $m^* = (\beta R/\alpha F)^{1/\mu} > 0$ of MCL parametrizes quasi-stationary states inside the corridor and stitching layers, and NHIM persistence guarantees the stratified fibration structure (Proposition 8.10) survives small perturbations. In this sense, ACG *completes* the dynamical picture initiated by MCL: MCL explains vertical confinement; ACG explains horizontal organization.

10.5 Relation to representation geometry and manifold learning

Manifold learning and representation geometry study the problem of recovering the intrinsic structure of a dataset from high-dimensional observations. Techniques such as diffusion maps, Isomap, and deep autoencoders seek low-dimensional representations that preserve local geometry and long-range topology of the underlying data manifold [18].

Points of contact. ACG's chart transitions are natural instances of representation geometry. Δ-lifting (Section 7, Proposition 8.6) recovers latent admissible connectivity (100% RISC recovery in the neutrino corpus) by mapping a fragmented chart to an admissible chart while preserving local gap ordering. This is structurally analogous to manifold unfolding: a non-linear reparametrization that removes chart-induced distortions. The deep-learning admissibility lift (Observation 7.8) is an explicit trained manifold map: the deep-L embedding recovers Full percolation (21/23 ladders, GR = 1.000) in the same way a representation model recovers the intrinsic manifold from compressed observations. The representational tower $\mathcal{R}_0 \to \mathcal{R}_1 \to \mathcal{R}_2 \to \mathcal{R}_3$ of the neutrino corpus is a concrete example of a progressive manifold refinement sequence.

Operational distinctions. ACG does not learn representations: its structural variables (GR, κ_{conn}, TD, n_{iso}) are computed deterministically from sorted gap vectors, not from data-driven embeddings. The connection to manifold learning operates in the opposite direction: ACG provides a *structural criterion* (admissibility via USL + PRP) against which different representational choices can be evaluated. A representation is "better" in the ACG sense if it places the source system deeper inside a Type-I basin rather than near a RISC artifact. Locality-preserving transforms (Δ-lifting, Conjecture 7.10) are a class of representation-space operations that ACG predicts will be admissibility-preserving; testing this class against known manifold learning methods is a natural direction for future work.

Takens-type context. The representation-covariant admissibility conjecture (Conjecture 7.10) has structural resonance with Takens' embedding theorem [19]: just as Takens guarantees that a delay-coordinate map generically recovers the attractor of a dynamical system, Conjecture 1 asserts that locality-preserving transforms recover the admissible cluster of the source process. The neutrino corpus provides the first operational evidence for this analogy.

10.6 Summary: structural position of ACG

ACG's structural position is that of a *unifying operational geometry* at the intersection of these frameworks. It is finer-grained than TDA (it resolves individual stitching junctions), more deterministic than percolation (gap vectors are fixed, not random), more protective than energy landscapes (no thermal escape from basins; MCL non-crossing applies), and more principled than generic manifold learning (the admissibility criterion provides a structural target for representation quality).

11 Toward a Global Geometry of Realizability Space

The convergence of metallic-glass chemistry, liquid-scintillator neutrino reconstruction, and protein folding dynamics—three domains with no chemical, physical, or biological overlap—into the same four basin types, the same stitching-defect mechanism, and the same recovery-by-chart-transition pattern suggests that the ACG is not domain-specific. It is a generic

Table 9: Structural correspondences between ACG and existing frameworks. Entries marked "partial" indicate shared structure with operational distinctions described in the text.

Framework	**ACG analogue**	**Key distinction**	**Alignment**
Classical percolation	κ_{conn} fixed points; GR threshold	Deterministic; localized; chart-dependent	Partial
Energy landscapes	Basins $\mathcal{A}_i$; corridors $\mathcal{K}_{ij}$; barriers	No scalar potential; connectivity geometry	Partial
Topological data analysis	Grid-invariant strata; barcode-like $\{S_j\}$	GR not Betti numbers; representation-sensitive	Partial
NHIM / Fenichel dynamics	Stratified fibration as internal NHIM structure	ACG adds internal topology to MCL's NHIM	Extends
Manifold learning	Δ-lifting; deep-L lift; chart transitions	Deterministic criterion; not data-driven	Partial
Takens embedding	Representation-covariant admissibility conjecture	Gap ordering vs. delay coordinates	Analogy

feature of realizability space under any heavy-tailed ordered-sequence representation.

Principle 11.1 (Merge-Boundary-as-Glue)**.** Admissible clusters form through, and are sustained by, recoverable bridge formation between spectral gaps. Systems with high bridge redundancy (oxides, dense glass compositions, high-stitching-strength states) maximize the width and stability of continuity corridors $\mathcal{K}_{ij}$, enabling robust admissible transport. Systems with low bridge redundancy (sulfides, fluorides, non-native representations) are susceptible to single-rung stitching failure, producing Type-III basin configurations that are nonetheless recoverable by chart transition or κ-extension.

We propose the following global picture of realizability space $\mathcal{R}$:

1. *Radial geometry*: $m(L)$ as scalar distance from $\partial\mathcal{M}_{\text{adm}}$.
2. *Basin topology*: κ_{conn}-stratified basin types (I–IV), corridors, and barriers as the internal angular geometry of $\mathcal{M}_{\text{adm}}$.
3. *Transport structure*: admissible trajectories within and between basins; three transport mechanisms (corridor bridging, chart transition, embedding lift); the bridgeability relation $\sim_{\mathcal{K}}$.
4. *Rigidity strata*: local stability regions Ω_L within each basin (BSR).
5. *Boundary dynamics*: FCC compression, tangential sliding, NHIM structure, and asymptotic non-crossability at $\partial\mathcal{M}_{\text{adm}}$ (MCL).

This five-layer description is the first complete geometric picture of realizability space offered by the UNNS program.

12 Predictions and Falsifiability

1. κ_{conn} **strata stability and corpus expansion.** The five currently observed κ_{conn} strata $\{0.562, 0.750, 1.000, 2.000, 10.000\}$ should persist under corpus expansion within the same normalized-ladder parametrization. New materials should fall into existing strata or reveal *additional* fixed-point strata, not populate a continuum of κ_{conn} values. Specifically: the Class II ($\kappa_{\text{conn}} = 0.750$) cluster is chemistry-independent and should admit members from any chemical family; the Class I-α ($\kappa_{\text{conn}} = 0.562$) cluster is

currently small (2 members) and may remain small—it represents an exceptionally dense sub-basin.

Falsification: discovery of κ_{conn} values stably intermediate between existing strata (e.g., $\kappa_{\text{conn}} = 0.65$ in multiple independent materials under the same IQR-scaling protocol), or failure of the grid-invariance property (Proposition 4.1) for new corpora evaluated under all seven configurations.

2. **Stitching-defect dominance.** In any corpus of heavy-tailed ordered sequences evaluated under IQR-scaled ε-neighborhoods with $\kappa_{\max} \approx 1$, the fraction of fragmented ladders with $n_{\text{iso}} = 1$ should exceed 75%.

   Falsification: a corpus where fragmented ladders show a broad n_{iso} distribution with $n_{\text{iso}} = 1$ comprising less than 50%.

3. **Δ-lifting recovery rate for RISC fragments.** For fragmented ladders arising from representation artifacts (classifier binning, orientation reversal, n-poverty in large-n settings), Δ-lifting should recover Full percolation in at least 90% of cases.

   Falsification: a corpus where Δ-lifting recovers fewer than 50% of fragmented ladders, indicating genuine structural discontinuities rather than representation artifacts dominate fragmentation.

4. **Oxide universality under normalization.** The 100% Full Continuity rate for normalized oxide ladders should persist under corpus expansion beyond the 69 tested oxides.

   Falsification: a normalized oxide ladder achieving GR < 1 under the standard 17-layer κ-grid.

5. **Basin-preserving versus barrier-crossing deformations.** Continuous deformations of a Type-I ladder should preserve the κ_{conn} stratum and remain within the Type-I basin. Passage from Type-I to Type-IV should require either an abrupt gap-outlier increase or a representation change.

   Falsification: a smooth continuous deformation of a Type-I ladder producing a Type-IV configuration without an identifiable representation artifact.

6. **Corridor-mediated recovery falls into a κ_{conn} fixed-point stratum.** For Type-III stitching-defect ladders ($n_{\text{iso}} = 1$, GR ≥ 0.95), the bridging connectivity threshold $\kappa^*_{\text{conn}} = g_{\max}/\sigma_{\text{IQR}}$ should fall into one of the existing fixed-point strata $\{0.562, 0.750, 1.000, 2.000, 10.000\}$ (Proposition 5.5) when the ladder is evaluated under the same IQR-scaling protocol. Equivalently, the ratio $g_{\max}/(\kappa^* \cdot \sigma_{\text{IQR}})$ should be approximately 1 for some κ^* in the fixed-point set.

   Falsification: a Type-III ladder with $n_{\text{iso}} = 1$ achieving full connectivity at κ^*_{conn} that is not within 5% of any established fixed point (e.g., $\kappa^*_{\text{conn}} = 1.43$ stably observed in multiple independent materials), indicating either a new attractor stratum or a breakdown of the fixed-point structure under the tested parametrization.

13 Conclusion

Prior UNNS manuscripts identified the admissibility boundary: where it lies, how systems approach it, and why they cannot cross it. The present manuscript identifies the internal topology of the admissible region itself.

The three new corpora converge on a consistent picture: admissible systems form coherent admissible basins organized by their κ-connectivity depth, connected by sparse continuity corridors, and separated from harder configurations by localized stitching barriers. The dominant fragmentation mechanism is not global structural collapse but a single admissibility stitch that fails to close—a binary threshold failure at the dominant tail gap.

The neutrino corpus demonstrates that apparent fragmentation is geometrically recoverable. All nine representation-induced fragmentation outcomes recover Full percolation under Δ-lifting (100% recovery rate), expanding FCC-like states from 5 to 34 instances. The giant component was latent beneath the representation artifact: structural admissibility is an intrinsic local-relational property of the source process, not a chart-dependent judgment.

The Admissible Cluster Geometry framework proposed here is the internal topology layer of realizability theory:

- The Margin-Confinement Law governs *whether* systems leave $\mathcal{M}_{\text{adm}}$.
- The ACG framework governs *how* systems are distributed and move within $\mathcal{M}_{\text{adm}}$.

Together they constitute the first complete geometric description of realizability space: boundary dynamics + internal basin topology + transport structure + rigidity strata = the full architecture of $\mathcal{M}_{\text{adm}}$.

The six most important results:

1. *Admissible basins are real, discrete, and grid-invariant.* The five κ_{conn} strata are structural fixed points of the admissibility manifold, not discretization artifacts.
2. *Fragmentation is overwhelmingly localized.* 80.4% of fragmented glass ladders have exactly one isolated node; the manifold remains globally coherent.
3. *Admissibility failure is a stitching threshold, not a bulk collapse.* The deciding variable is whether the dominant tail gap exceeds $\kappa_{\max} \cdot \sigma_{\text{IQR}}$; bulk TD differences are sub-percent.
4. *Recoverable connectivity is a geometric property.* All representation-induced fragmentation is resolvable by locality-preserving chart transition; the source retains latent continuity.
5. *Admissible clusters are defined by basin bridgeability.* The basin bridgeability relation $\approx_{\mathcal{K}}$ (Definition 7.3) partitions basins into admissible clusters at the basin level; the ladder bridgeability relation $\sim_{\mathcal{K}}$ (Definition 5.8) generates intra-basin ladder-level connectivity.
6. *The internal topology of $\mathcal{M}_{\text{adm}}$ is a stratified fibration.* The operator T generates discrete κ-strata; $\sim_{\mathcal{K}}$ generates cluster fibers; π gives the radial projection. Together with the MCL's vertical boundary protection, these constitute the complete coordinate system for realizability space (Proposition 8.10).

All findings are scoped to the tested corpora. The ACG framework holds within the UNNS axiomatic framework under the corpus-specific conditions of Section 4. Generalization beyond the tested corpora requires additional empirical support.

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